$$r(\rho,\theta) = \begin{cases} y = r\sin(\theta) & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ z = \sqrt{r^2-\rho^2} \end{cases}$$

$$r_\rho = \left(\cos(\theta),\ \sin(\theta),\ -\frac{\rho}{\sqrt{r^2-\rho^2}}\right)$$

$$r_\theta = \left(-\rho\sin(\theta),\ \rho\cos(\theta),\ 0\right)$$

La Jacobiana è:

$$J(r) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & -\frac{\rho}{\sqrt{r^2-\rho^2}} \\ -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{pmatrix}$$

$$r_\rho \wedge r_\theta = \left(\frac{\rho^2\cos(\theta)}{\sqrt{r^2-\rho^2}},\ \frac{\rho^2\sin(\theta)}{\sqrt{r^2-\rho^2}},\ \rho\right)$$

$$\|\dots\| = \sqrt{\frac{\rho^4\cos^2(\theta)}{r^2-\rho^2} + \frac{\rho^4\sin^2(\theta)}{r^2-\rho^2} + \rho^2} =$$

$$= \sqrt{\frac{\rho^4+\rho^2r^2-\rho^4}{r^2-\rho^2}} = \sqrt{\frac{\rho^2r^2}{r^2-\rho^2}} = \frac{\rho r}{\sqrt{r^2-\rho^2}}$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{r\cos(\theta)} \frac{\rho r}{\sqrt{r^2-\rho^2}}\,d\rho\,d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} -r\sqrt{r^2-\rho^2}\,\Big|_0^{r\cos(\theta)}\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} -r\left(\sqrt{r^2-r^2\cos^2(\theta)} - r\right)d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} -r\left(\sqrt{r^2(1-\cos^2(\theta))} - r\right)d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} -r\left(r|\sin(\theta)| - r\right)d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} -r^2|\sin(\theta)| + r^2\,d\theta =$$

$$= \int_{-\frac{\pi}{2}} \ldots \int_{-\frac{\pi}{2}}$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} r^2 \left(1 - |\sin(\theta)|\right) d\theta = r^2 \int_{-\frac{\pi}{2}}^{\pi/2} 1 - |\sin(\theta)|\, d\theta =$$

$$= 2 r^2 \int_0^{\pi/2} 1 - \sin(\theta)\, d\theta = 2r^2 \left(\theta + \cos(\theta)\right)\Big|_0^{\pi/2} =$$

$$= 2 r^2 \left(\frac{\pi}{2} + 0 - 1\right) = \pi r^2 - 2 r^2 = r^2(\pi - 2)$$

b): Le equazioni parametriche del bordo γ si ottengono ponendo $\rho = r\cos(\theta)$

$$\Rightarrow \quad \begin{cases} x = r\cos^2(\theta) \\ y = r\sin(\theta)\cos(\theta) \\ z = r\sqrt{1-\cos^2(\theta)} = r\,|\sin(\theta)| \end{cases}$$

Al variare di θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ si ha che $z = z(\theta)$ non è derivabile per $\theta = 0$ $\Rightarrow$ γ non è C^1

c):
$$\begin{cases} F_1 = x^2 + y^2 + z^2 - r^2 = 0 \\ F_2 = x^2 - r x + y^2 = 0 \end{cases}$$

Jacobiano:
$$\begin{pmatrix} 2x & 2y & 2z \\ 2x - r & 2y & 0 \end{pmatrix}$$

Consideriamo:

$$\begin{vmatrix} 2x & 2y \\ 2x-r & 2y \end{vmatrix} = 4xy - (2y\cdot(2x-r)) =$$
$$= 4xy - (4xy - 2yr) = 2ry$$

$$\begin{vmatrix} 2x & 2z \\ 2x-r & 0 \end{vmatrix} = 0 - (2z(2x-r)) = -(4xz - 2rz) =$$
$$= 2rz - 4xy$$

$$\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix} = -4yz$$

ci danno fastidio i punti del tipo $y=0,\ z=0$,

cioè $(x,0,0)$

Però $(x,0,0)$ con $x=r \Rightarrow (r,0,0) \in \gamma$

$\Rightarrow$ Non è verificato il teorema del DINI $\blacksquare$

Es:

Calcolare l'area della superficie del solido definito da:

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ \middle|\ x^2+z^2 \leq 2\ ,\ 0 \leq y \leq 5-2x \right\}$$

SOL

Scompongo il solido in tre superfici:

$$\Sigma_1 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 \leq 2, \quad y = 0 \right\}$$

$$\Sigma_2 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 \leq 2, \quad y = 5 - 2x \right\}$$

$$\Sigma_3 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 = 2, \quad 0 \leq y \leq 5 - 2x \right\}$$

Parametrizzo Σ_1, Σ_2, Σ_3

$$\Psi(x,z) : \begin{cases} x = x \\ y = 0 \\ z = z \end{cases}, \quad \varphi(x,z) : \begin{cases} x = x \\ y = 5 - 2x \\ z = z \end{cases}, \quad \Gamma(\theta, y) : \begin{cases} x = \sqrt{2}\cos(\theta) \\ y = y \\ z = \sqrt{2}\sin(\theta) \end{cases}$$

$\Psi_x = (1,0,0) \qquad \varphi_x = (1,-2,0) \qquad \Gamma_\theta = (-\sqrt{2}\sin(\theta), 0, \sqrt{2}\cos(\theta))$

$\Psi_z = (0,0,1) \qquad \varphi_z = (0,0,1) \qquad \Gamma_y = (0, 1, 0)$

$\Psi_x \wedge \Psi_z = (0,-1,0) \qquad \varphi_x \wedge \varphi_z = (-2,-1,0) \qquad \Gamma_\theta \wedge \Gamma_y = (-\sqrt{2}\cos(\theta), 0, -\sqrt{2}\sin(\theta))$

$\| \dots \| = 1 \qquad \| \dots \| = \sqrt{5} \qquad \| \dots \| = \sqrt{2}$

$$\int_{\Sigma_1} d\sigma = \int_D dx\, dz =$$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [0, \sqrt{2}] \\ z = \rho\sin(\theta) & \theta \in [0, 2\pi] \end{cases}$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [0,\sqrt{2}] \\ z = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$

$$\Rightarrow \int_0^{2\pi}\int_0^{\sqrt{2}} \rho \, d\rho \, d\theta = \int_0^{2\pi} \frac{\rho^2}{2}\Big|_0^{\sqrt{2}} d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$$

$$\int_{\Sigma_2} d\sigma = \int_D \sqrt{5} \, dx\, dz = \sqrt{5}\int_D dx\, dz$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,\sqrt{2}] \\ z = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \sqrt{5}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho \, d\rho \, d\theta = \sqrt{5}\cdot\int_0^{2\pi} \frac{\rho^2}{2}\Big|_0^{\sqrt{2}} d\theta = 2\pi\sqrt{5}$$

$$\int_{\Sigma_3} d\sigma = \int_D \sqrt{2} \, dy\, d\theta = \sqrt{2}\int_D dy\, d\theta = \sqrt{2}\int_0^{2\pi}\int_0^{5-2\sqrt{2}\cos(\theta)} dy\, d\theta =$$

$$= \sqrt{2}\int_0^{2\pi} 5 - 2\sqrt{2}\cos(\theta) \, d\theta =$$

$$= \sqrt{2}\left(5\theta - 2\sqrt{2}\sin(\theta)\right)\Big|_0^{2\pi} =$$

$$= \sqrt{2}\,(10\pi) = 10\pi\sqrt{2}$$

Sommiamo $\int_{\Sigma_1} + \int_{\Sigma_2} + \int_{\Sigma_3} = 2\pi + 2\pi\sqrt{5} + 10\pi\sqrt{2} =$

$= 2\pi(1 + \sqrt{5} + 5\sqrt{2})$

$\blacksquare$

ES:

Si consideri il cilindro circolare retto avente per base nel piano x, z la circonferenza di centro l'origine e raggio 2

Si calcoli l'area della porzione di superficie del cilindro contenuta nel primo ottante ($x \geq 0, y \geq 0, z \geq 0$) e nel semispazio di equazione $5x > y$

SOL

$$C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 \leq 2,\; 0 \leq y < 5x,\; \begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix} \right\}$$

Scompongo in tre superfici:

$$\Sigma_1 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 \leq 4,\; y = 0,\; \begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix} \right\}$$

$$\Sigma_2 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + z^2 \leq 4,\; y = 5x,\; \begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix} \right\}$$

$$\Sigma_3 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+z^2=4 \,,\; 0 \le y < 5x \right\}$$

Parametrizzo $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\varphi(x,z) = \begin{cases} x = x \\ y = 0 \\ z = z \end{cases} \;,\quad \psi(x,z) = \begin{cases} x = x \\ y = 5x \\ z = z \end{cases} \;,\quad \Gamma(\theta, y) = \begin{cases} x = 2\cos(\theta) \\ y = y \\ z = 2\sin(\theta) \end{cases}$$

$\varphi_x = (1,0,0)$ $\quad$ $\psi_x = (1,5,0)$ $\quad$ $\Gamma_\theta = (-\sqrt{2}\sin(\theta),\ 0,\ \sqrt{2}\cos(\theta))$

$\varphi_z = (0,0,1)$ $\quad$ $\psi_z = (0,0,1)$ $\quad$ $\Gamma_y = (\ 0\ ,1,\ 0\ )$

$\varphi_x \wedge \varphi_z = (0,-1,0)$ $\quad$ $\psi_x \wedge \psi_z = (5,-1,0)$ $\quad$ $\Gamma_\theta \wedge \Gamma_y = (-2\cos(\theta), 0, -2\sin(\theta))$

$\|\ldots\| = 1$ $\quad$ $\|\ldots\| = \sqrt{26}$ $\quad$ $\|\ldots\| = 2$

$$\int_{\Sigma_1} d\sigma = \int_D dx\,dz =$$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [0,2] \\ z = \rho\sin(\theta) & \theta \in [0,\frac{\pi}{2}] \end{cases}$

$$\Rightarrow \int_0^{\pi/2}\int_0^2 \rho\, d\rho\, d\theta = \int_0^{\pi/2} \frac{\rho^2}{2}\Big|_0^2 = \pi$$

$$\cdot \int_{\Sigma_2} d\sigma = \int_D \sqrt{26}\, dx\,dz = \sqrt{26}\int_D dx\,dz = \sqrt{26}\int_0^{\pi/2}\int_0^2 \rho\, d\rho\, d\theta =$$

$$= \sqrt{26}\int_0^{\pi/2} \frac{\rho^2}{2}\Big|_0^2 d\theta = \sqrt{26}\,\pi$$

$$= \sqrt{26} \int_0^{\pi/2} \frac{\rho^2}{2} \Big|_0^{2} \, d\theta = \sqrt{26}\, \pi$$

$$\cdot \int_{\Sigma_3} d\sigma = \int_D 2 \, d\theta \, dz = 2 \int_0^{\pi/2} \int_0^{10\cos(\theta)} dz \, d\theta =$$

$$= 2 \int_0^{\pi/2} 10 \cos(\theta) \, d\theta = 2 \left(10 \sin(\theta) \right) \Big|_0^{\pi/2} =$$

$$= 2 \left(10 \right) = 20$$

$$\Rightarrow \int_{\Sigma_1} d\sigma + \int_{\Sigma_2} d\sigma + \int_{\Sigma_3} d\sigma = \frac{\pi}{2} + \frac{\sqrt{26}}{2}\pi + 20$$

$\blacksquare$

Es:

Determinare l'equazione del cono che dal punto di coordinate $(1,0,2)$, proietta la curva di equazioni

$$\begin{cases} z = 0 \\ (x-1)^2 + y^2 = 1 \end{cases}$$

Calcolare inoltre l'area della porzione di superficie

Calcolare inoltre l'area della porzione di superficie di tale cono delimitata dalle condizioni

$$x \geq \frac{3}{2} \quad , \quad 0 \leq z \leq 1$$

SOL

L'equazione del cono si ottiene:

$$(z-z_0)^2 = \text{costante} \cdot \left[(x-x_0)^2 + (y-y_0)^2\right]$$

$$\Rightarrow \quad (z-2)^2 = 4\left[(x-1)^2 + y^2\right]$$

$$\Rightarrow \quad (z-2)^2 = 4(x-1)^2 + 4y^2$$

Scompongo in tre superfici:

$$\Sigma_1 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z=0 \;,\; (x-1)^2 + y^2 \leq 1 \;,\; x \geq \frac{3}{2} \right\}$$

$$\Sigma_2 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z=1 \;,\; (x-1)^2 + y^2 \leq \frac{1}{4} \;,\; x \geq \frac{3}{2} \right\}$$

$$\Sigma_3 : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underline{(z-2)^2 = 4(x-1)^2 + 4y^2} \;,\; 0 \leq z \leq 1 \;,\; x \geq \frac{3}{2} \right\}$$

$$" \; \frac{(z-2)^2}{4} = (x-1)^2 + y^2 \Rightarrow \rho = \frac{z-2}{2}$$

Parametrizzo $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = 0 \end{cases} \;,\quad \psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1 \end{cases} \;,\quad \Gamma(\theta, z) = \begin{cases} x = 1 + \frac{z-2}{2}\cos(\theta) \\ y = \frac{z-2}{2}\sin(\theta) \\ z = z \end{cases}$$

$$\varphi_x = (1,0,0) \qquad \psi_x = (1,0,0) \qquad \Gamma_\theta = \left(-\frac{z-2}{2}\sin(\theta),\; \frac{z-2}{2}\cos(\theta),\; 0\right)$$

$$\varphi_x = (1,0,0) \qquad \varphi_x = (1,0,0) \qquad \Gamma_\theta = \left(-\frac{z-2}{2}\sin(\theta),\ \frac{z-2}{2}\cos(\theta),\ 0\right)$$

$$\varphi_y = (0,1,0) \qquad \varphi_y = (0,1,0) \qquad \Gamma_z = \left(\frac{1}{2}\cos(\theta),\ \frac{1}{2}\sin(\theta),\ 1\right)$$

$$\varphi_x \wedge \varphi_y = (0,0,1) \qquad \varphi_x \wedge \varphi_y = (0,0,1) \qquad \Gamma_\theta \wedge \Gamma_z = \left(\frac{z-2}{2}\cos(\theta),\ \frac{z-2}{2}\sin(\theta),\ -\frac{z-2}{4}\right)$$

$$\|\ldots\| = 1 \qquad \|\ldots\| = 1 \qquad \|\ldots\| = \sqrt{\left(\frac{z-2}{2}\right)^2 + \left(\frac{z-2}{4}\right)^2}$$

$$= \sqrt{\frac{z^2+4-4z}{4} + \frac{z^2+4-4z}{16}} =$$

$$= \sqrt{\frac{4z^2+16-16z+z^2+4-4z}{16}} =$$

$$= \frac{1}{4}\sqrt{5z^2+20-20z} = \frac{1}{4}(\sqrt{5}z - \sqrt{20})$$

$$\int_{\Sigma_1} d\sigma = \int_D dx\,dy$$

in polari:

$$\begin{cases} x = 1 + \rho\cos(\theta) & \rho \in [\frac{1}{2}, 1] \\ y = \rho\sin(\theta) & \theta \in [-\frac{\pi}{3}, \frac{\pi}{3}] \end{cases}$$

$$(x-1)^2 + y^2 \leq 1$$

$$\Rightarrow (x-1)^2 \leq 1 - y^2$$

$$\Rightarrow x - 1 \leq \sqrt{1-y^2}$$

$$\Rightarrow \frac{3}{2} < x < 1 + \sqrt{1-y^2}$$

$$\Downarrow$$

$$\frac{3}{2} < 1 + \rho\cos(\theta) < 1 + \sqrt{1-\rho^2\sin^2(\theta)}$$

$$\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \Leftarrow \begin{cases} \frac{1}{2} < \rho\cos(\theta) < \sqrt{1-\rho^2\sin^2(\theta)} \\ \frac{1}{4} < \rho^2\cos^2(\theta) < 1 - \rho^2\sin^2(\theta) \end{cases}$$

$$\frac{1}{4} < \rho^2 \cos^2(\theta) + \rho^2 \sin^2(\theta) < 1$$

$$\frac{1}{4} < \rho^2 < 1 \quad \Rightarrow \quad \frac{1}{2} < \rho < 1$$

$$\Rightarrow \int_{\frac{1}{2}}^{1} \int_{-\frac{\pi}{3}}^{\pi/3} \rho \, d\theta \, d\rho = \frac{2}{3}\pi \int_{\frac{1}{2}}^{1} \rho \, d\rho = \frac{2}{3}\pi \left(\frac{\rho^2}{2} \right) \Big|_{\frac{1}{2}}^{1} =$$

$$= \frac{2}{3}\pi \left(\frac{1}{2} - \frac{1}{8} \right) = \frac{2}{3}\pi \left(\frac{3}{8} \right) = \frac{\pi}{4}$$

$$\int_{\Sigma_2} d\sigma = \int_D dx \, dy =$$

in polari:

$$\begin{cases} x = 1 + \rho \cos(\theta) \\ y = \rho \sin(\theta) \end{cases}$$

$$(x-1)^2 + y^2 \leq \frac{1}{4} \qquad , \quad x > \frac{3}{2}$$

$$\Rightarrow \quad (x-1)^2 \leq \frac{1}{4} - y^2$$

$$\Rightarrow \quad x - 1 \leq \sqrt{\frac{1}{4} - y^2}$$

$$\Rightarrow \quad \frac{3}{2} < x < 1 + \sqrt{\frac{1}{4} - y^2}$$

$$\Downarrow$$

$$\frac{3}{2} < 1 + \rho \cos(\theta) < 1 + \sqrt{\frac{1}{4} - \rho^2 \sin^2(\theta)}$$

$$\Rightarrow \quad \frac{1}{2} < \rho \cos(\theta) < \sqrt{\frac{1}{4} - \rho^2 \sin^2(\theta)}$$

$$\frac{1}{4} < \rho^2 \cos^2(\theta) < \frac{1}{4} - \rho^2 \sin^2(\theta)$$

$$\frac{1}{4} < \rho^2 < \frac{1}{4} \Rightarrow \rho = \frac{1}{2}$$

$$\Rightarrow \int_{\Sigma_2} d\sigma = 0$$

$$\int_{\Sigma_3} d\sigma = \int_D \frac{1}{4}(z\sqrt{5} - \sqrt{20})\, d\theta\, dz$$

$$x = 1 + \frac{z-2}{2}\cos(\theta) \quad , \quad x > \frac{3}{2} \quad , \quad z \in [0,1]$$

$$\Rightarrow \frac{3}{2} < 1 + \frac{z-2}{2}\cos(\theta)$$

$$\Rightarrow \frac{1}{2} < \frac{z-2}{2}\cos(\theta) \Rightarrow 1 < (z-2)\cos(\theta)$$

Se $z=0 \Rightarrow -\frac{1}{2} < \cos(\theta)$

Se $z=1 \Rightarrow -1 < \cos(\theta)$

$$\Rightarrow \cos(\theta) > -\frac{1}{2} > -1 \Rightarrow \cos(\theta) > -\frac{1}{2} \Rightarrow$$ se $\theta = \frac{2}{3}\pi$ $\Rightarrow \cos(\frac{2}{3}\pi) = -\frac{1}{2}$

$$\Rightarrow \theta \in \left[\frac{4}{3}\pi , \frac{2}{3}\pi\right]$$

$$\Rightarrow \int_0^1 \int_{\frac{4}{3}\pi}^{\frac{2}{3}\pi} \frac{1}{4}(z\sqrt{5} - \sqrt{20}) = \int_0^1 \int_{\frac{4}{3}\pi}^{\frac{2}{3}\pi} \frac{\sqrt{5}}{4}z - \frac{\sqrt{20}}{4}\, d\theta\, dz =$$

$$= -\frac{2}{3}\pi \int_0^1 \frac{\sqrt{5}}{4}z - \frac{\sqrt{20}}{4}\, dz = -\frac{2}{3}\pi\left(\frac{\sqrt{5}}{4}\frac{z^2}{2} - \frac{\sqrt{20}}{4}z\right)\Big|_0^1 =$$

$$= -\frac{2}{3}\pi\left(\frac{\sqrt{5}}{8} - \frac{\sqrt{20}}{4}\right) = -\frac{2}{3}\pi\left(\frac{\sqrt{5} - 2\sqrt{20}}{8}\right) = -\frac{\sqrt{5} - 2\sqrt{20}}{12}\pi = -\frac{\sqrt{5}}{12}\pi + \frac{\sqrt{5}}{3}\pi$$

Sommo i tre integrali:

$$\int_{\Sigma_1} d\sigma + \int_{\Sigma_2} d\sigma + \int_{\Sigma_3} d\sigma = \frac{\pi}{4} + 0 + \frac{\sqrt{5}}{3}\pi - \frac{\sqrt{5}}{12}\pi$$

$$= \frac{1}{4}\pi + \frac{\sqrt{5}}{3}\pi - \frac{\sqrt{5}}{12}\pi = \frac{3\pi + 4\sqrt{5}\pi - \sqrt{5}\pi}{12} =$$

$$= \frac{3\pi + 3\pi\sqrt{5}}{12} = \frac{\pi + \pi\sqrt{5}}{4} = \pi\left(1 + \frac{\sqrt{5}}{4}\right)$$

$\blacksquare$

Esercizio:

Calcolare l'integrale curvilineo di forma differenziale

$$\int_{\partial S^+} dx + y\,dz$$

dove S è la superficie di equazione cartesiana

$$z = x + 3\sin(x^2 - y^2) \quad , \quad x^2 + y^2 \leq 1$$

Sol

$$\int_{\partial S^+} \omega = \int_S \langle \mathrm{rot}(F), n \rangle = \int_A F \cdot n$$

$F = (1, 0, y)$

Parametrizzo S:

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = x + 3\sin(x^2 - y^2) \end{cases}$$

$\psi_x = (1, 0, 1 + 6x\cos(x^2 - y^2))$

$\psi_y = (0, 1, -6y\cos(x^2 - y^2))$

[illegible]

$$\psi_y = (0, 1, -6y\cos(x^2-y^2))$$

$$\psi_x \wedge \psi_y = (-1-6x\cos(x^2-y^2),\ 6y\cos(x^2-y^2),\ 1)$$

$$\int_A F\cdot n = \int_A \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} \begin{pmatrix} -1-6x\cos(x^2-y^2) \\ 6y\cos(x^2-y^2) \\ 1 \end{pmatrix} dx\,dy$$

$$= \int_A -1-6x\cos(x^2-y^2)+y \; dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} -1-6x\cos(x^2-y^2)+y \; dx\,dy =$$

$$= \int_{-1}^{1} \left. -x-\cancel{3\sin(x^2-y^2)}+xy \right|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy =$$

$$= \int_{-1}^{1} -2\sqrt{1-y^2}+2y\sqrt{1-y^2} \; dy = \ldots = -\pi$$

E_2 :

Calcolare l'integrale curvilineo di

$$\omega = z^2\,dx + xyz\,dy + xz^2\,dz$$

estesa al bordo ∂S^+ della superficie laterale del solido B (porzione di superficie cilindrica)

(a) $x + \frac{y}{2} + z = 3 \Rightarrow z = 3 - x - \frac{y}{2}$

Sol

$x+\frac{y}{2}+z=3 \Rightarrow z=3-x-\frac{y}{2}$

$F=(z^2, xyz, xz^2)$

$$\int_{\partial S} w = \int_S rot(F)\cdot n \, d\sigma$$

$$rot(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ z^2 & xyz & xz^2 \end{vmatrix} = (-xy, 2z-z^2, yz)$$

Parametrizzo S:

$$\psi(\theta,z) = \begin{cases} x=\cos(\theta) \\ y=\sin(\theta) \\ z=z \end{cases} \qquad \begin{array}{l} \theta\in[0,2\pi] \\ z\in[0, 3-x-\frac{y}{2}] \\ \Rightarrow z\in[0, 3-\cos(\theta)-\frac{\sin(\theta)}{2}] \end{array}$$

$\psi_\theta = (-\sin(\theta), \cos(\theta), 0)$

$\psi_z = (0, 0, 1)$

$\psi_\theta \wedge \psi_z = (\cos(\theta), \sin(\theta), 0)$

$$\Rightarrow \int_S rot(F)\cdot n = \int_A rot(F(\psi))\cdot N = \int_A \begin{pmatrix} -\cos(\theta)\sin(\theta) \\ 2z-z^2 \\ z\sin(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

$$= \int_A -\sin(\theta)\cos^2(\theta) + (2z-z^2)\sin(\theta) \, d\theta \, dz =$$

$$= \int_0^{2\pi}\int_0^{3-\cos(\theta)-\frac{\sin(\theta)}{2}} -\sin(\theta)\cos^2(\theta) + 2z\sin(\theta) - z^2\sin(\theta) \, dz \, d\theta =$$

$$= \dots = \frac{\pi}{8}$$

$$= \dots = \frac{\pi}{8}$$

$\blacksquare$

17. Simulazione esame 2

giovedì 26 gennaio 2023 17:10

Es 1 :

Dire se $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 = 4r^2,\ x^2+y^2-2rx = 0 \right\}$

è varietà in $\mathbb{R}^3$, con $r>0$

Trovare i punti di M che hanno min distanza da $P = (0, 2r, 0)$

SOL

$F_1 = x^2+y^2+z^2-4r^2$

$F_2 = x^2+y^2-2rx$

$$\begin{pmatrix} 2x & 2y & 2z \\ 2x-r & 2y & 0 \end{pmatrix}$$

$$\begin{vmatrix} 2x & 2y \\ 2x-r & 2y \end{vmatrix} = 4xy-(4xy-2yr) = 2yr$$

$$\begin{vmatrix} 2x & 2z \\ 2x-r & 0 \end{vmatrix} = -4xz+2zr = z(2r-4x)$$

$$\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix} = -4zy$$

+ da finire*

ES 2 :

$\gamma(t) = (4\cos(t) - \cos(4t),\ 4\sin(t) - \sin(4t))$

$t \in [0, 2\pi]$

Calcolare la misura di A

SOL

$$\iint_A dx\,dy = \frac{1}{2}\int_{\partial A} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4\cos(t)-\cos(4t)\right)\left(4\cos(t)-4\cos(4t)\right) - \left(4\sin(t)-\sin(4t)\right)\left(-4\sin(t)+4\sin(4t)\right)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 16\cos^2(t) - 16\cos(t)\cos(4t) - 4\cos(t)\cos(4t) + 4\cos^2(4t) + 16\sin^2(t) - 16\sin(t)\sin(4t) - 4\sin(t)\sin(4t) + 4\sin^2(4t)$$

$$= \frac{1}{2}\int_0^{2\pi} 20 - 20\underbrace{\cos(t)\cos(4t)}_{\Downarrow\; 4\cos^3(t)-3\cos(t)} - 20\underbrace{\sin(t)\sin(4t)}_{\Downarrow\; ???}\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 20\, dt = 20\pi$$

$\blacksquare$

Es 3:

$$F(x,y,z) = \left(z-y,\ x(1+x^2),\ xy\right)$$

Calcolare flusso del rotore di F di Σ

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = 1 - \left(\frac{x^2}{4}+\frac{y^2}{9}\right),\ \frac{x^2}{4}+\frac{y^2}{9} < 1 \right\}$$

Sol

$$\int_\Sigma \text{rot}(F)\cdot \hat{n}\, d\sigma = \int_{\partial\Sigma} F$$

$$= \int_{\partial\Sigma} z-y\, dx + x(1+z^2)\, dy + xy\, dz$$

x in coordinate ellittiche:

$$\begin{cases} x = 2\cos(\theta) \\ y = 3\sin(\theta) \\ z = 0 \end{cases} \qquad \theta \in [0, 2\pi]$$

$$\int_0^{2\pi} 6\sin^2(\theta) + 6\cos^2(\theta)\, d\theta = 12\pi$$

$\blacksquare$

Es 4:

Studiare irrazionalità ed esatezza di

$$F(x,y) = \left(\frac{y^2+2xy+ax^2}{(x^2+y^2)^2} \; , \; - \frac{x^2+2xy+by^2}{(x^2+y^2)^2} \right)$$

al variare di a e $b \in \mathbb{R}$

Sol

$$\frac{\partial}{\partial y} F_1 = \frac{(2y+2x)(x^2+y^2)^2 - 4y(y^2+2xy+ax^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial}{\partial x} F_2 = \frac{(-2x-2y)(x^2+y^2)^2 + 4x(x^2+2xy+by^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2 \Rightarrow$$

$$(2y+2x)(x^2+y^2) - 4y(y^3+2xy+ax^2) = (-2x-2y)(x^2+y^2) + 4x(x^2+2xy+by^2)$$

$\Rightarrow$

$$-4x^2y(1+a) - 4xy^2(1+b) = 0$$

$$\Rightarrow x(1+a) + y(1+b) = 0$$

$$\Rightarrow \begin{array}{l} a = -1 \\ b = -1 \end{array}$$

ESATEZZA:

$$\gamma : \begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \end{cases} \qquad \theta \in [0, 2\pi]$$

$$\int_\gamma \frac{y^2+2xy-x^2}{(x^2+y^2)^2}\, dx - \frac{x^2+2xy-y^2}{(x^2+y^2)^2}\, dy$$

$$\Rightarrow \int_0^{2\pi} \left(\sin^2(\theta) + 2\sin(\theta)\cos(\theta) - \cos^2(\theta)\right)\left(-\sin(\theta)\right) - \cos(\theta)\left(\cos^2(\theta) + 2\sin(\theta)\cos(\theta) - \sin^2(\theta)\right)$$

$$\Rightarrow \int_0^{2\pi} -\sin^3(\theta) - 2\sin^2(\theta)\cos(\theta) + \sin(\theta)\cos^2(\theta) - \cos^3(\theta) - 2\sin(\theta)\cos^2(\theta) + \sin^2(\theta)\cos(\theta)$$

$$\Rightarrow \int_0^{2\pi} -\sin^3(\theta) - \cos^3(\theta)\, d\theta = \int_0^{2\pi} -\cos^3(\theta)\, d\theta = 0$$

$\Rightarrow$ ESATTA

$\blacksquare$

ES 6 :

Calcolare : $$\iiint_\Omega e^{\frac{x+y}{\sqrt{2}}}\, dx\, dy\, dz$$

dove $\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} < 1 \right\}$

SOL

* Difficile *

$\blacksquare$

18. Prova mandata da Arianna

venerdì 27 gennaio 2023 17:27

Es 1:

Per quali valori di α, β la forma differenziale

$$w = \left(2x\sin(y) + \alpha \frac{y}{(x-1)^2+y^2}\right)dx + \left(\beta x^2\cos(y) - \alpha \frac{x-1}{(x-1)^2+y^2}\right)dy$$

è chiusa in $\mathbb{R}^2 \setminus \{(1,0)\}$

Determinare i parametri per cui è esatta e trovare una primitiva

SOL

$$\frac{\partial}{\partial y} F_1 \overset{?}{=} \frac{\partial}{\partial x} F_2$$

$$\frac{\partial}{\partial y} F_1 = 2x\cos(y) + \frac{\alpha((x-1)^2+y^2) - \alpha y(2y)}{((x-1)^2+y^2)^2}$$

$$= 2x\cos(y) + \frac{\alpha(x-1)^2 + \alpha y^2 - 2\alpha y^2}{((x-1)^2+y^2)^2}$$

$$\frac{\partial}{\partial x} F_2 = 2\beta x\cos(y) + \frac{-\alpha((x-1)^2+y^2) - (-\alpha x+\alpha)(2)(x-1)}{((x-1)^2+y^2)^2}$$

$$= 2\beta x \cos(\eta) + \frac{-\alpha(x-1)^2 - \alpha\eta^2 - 2(x-1)(-\alpha x + \alpha)}{\left((x-1)^2 + \eta^2\right)^2}$$

$$= 2\beta x \cos(\eta) + \frac{-\alpha(x-1)^2 - \alpha\eta^2 - 2\left(-\alpha x^2 + \alpha x + \alpha x - \alpha\right)}{\left((x-1)^2 + \eta^2\right)^2}$$

$$= 2\beta x \cos(\eta) + \frac{-\alpha(x-1)^2 - \alpha\eta^2 + 2\alpha x^2 + 2\alpha}{\left((x-1)^2 + \eta^2\right)^2}$$

$$= 2\beta x \cos(\eta) + \frac{-\alpha(x^2 + 1 - 2x) - \alpha\eta^2 + 2\alpha x^2 + 2\alpha}{\left((x-1)^2 + \eta^2\right)^2}$$

$$= 2\beta x \cos(\eta) + \frac{-\alpha x^2 - \alpha + 2\alpha x - \alpha\eta^2 + 2\alpha x^2 + 2\alpha}{\left((x-1)^2 + \eta^2\right)^2}$$

$$= 2\beta x \cos(\eta) + \frac{\alpha(-x^2 - 1 + 2x - \eta^2 + 2x^2 + 2)}{\left((x^2-1)^2 + \eta^2\right)^2} =$$

$$= 2\beta x \cos(\eta) + \frac{\alpha(x^2 + 2x + 1 - \eta^2)}{\left((x-1)^2 + \eta^2\right)^2}$$

$$\Rightarrow \frac{\partial}{\partial \eta} F_1 = \frac{\partial}{\partial x} F_2$$

$$\Rightarrow \frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$$

$$\Rightarrow 2x\cos(y) + \frac{\alpha(x-1)^2 + \alpha y^2 - 2\alpha y^2}{\left((x-1)^2+y^2\right)^2} = 2\beta x\cos(y) + \frac{\alpha(x^2+2x+1-y^2)}{\left((x-1)^2+y^2\right)^2}$$

$$\Rightarrow$$

$$2x\cos(y) + \frac{\alpha\left((x-1)^2+y^2-2y^2\right)}{\left((x-1)^2+y^2\right)^2} = 2\beta x\cos(y) + \frac{\alpha(x^2+2x+1-y^2)}{\left((x-1)^2+y^2\right)^2}$$

$$\Rightarrow$$

$$2x\cos(y) + \frac{\alpha(x^2+1-2x-y^2)}{\left((x-1)^2+y^2\right)^2} = 2\beta x\cos(y) + \frac{\alpha(x^2+2x+1-y^2)}{\left((x-1)^2+y^2\right)^2}$$

$$\Rightarrow$$

$$2x\cos(y) - 2x\beta\cos(y) + \frac{\alpha(x^2+1-2x-y^2)}{\left((x-1)^2+y^2\right)^2} - \frac{\alpha(x^2+2x+1-y^2}{\left((x-1)^2+y^2\right)^2} = 0$$

$$\Rightarrow$$

$$2x\cos(y)(1-\beta) + \frac{\cancel{\alpha x^2}+\cancel{\alpha}-2\alpha x-\cancel{\alpha y^2}-\cancel{\alpha x^2}-2\alpha x-\cancel{\alpha}+\cancel{\alpha y^2}}{\left((x-1)^2+y^2\right)^2} = 0$$

$$\Rightarrow$$

$$2x\cos(y)(1-\beta) + \frac{-4\alpha x}{\left((x-1)^2+y^2\right)^2} = 0 \quad \Rightarrow \begin{cases} \beta = 1 \\ \alpha = 0 \end{cases}$$

$\Rightarrow$ chiusa

$\Rightarrow \omega = 2x \sin(y)\,dx + x^2 \cos(y)\,dy$

$U(x,y) = \int 2x \sin(y)\,dx = x^2 \sin(y) + c(y)$

$U_y = x^2 \cos(y) + c'(y)$

$\Rightarrow U_y = F_2 \Rightarrow x^2 \cos(y) + c'(y) = x^2 \cos(y)$

$\Rightarrow c'(y) = 0 \Rightarrow c(y) = K$

$\Rightarrow U(x,y) = x^2 \sin(y) + K$

$\exists\, U(x,y) \Rightarrow \omega$ è esatta

$\blacksquare$

Es 3

Calcolare flusso del campo $F(x,y,z) = (x^4, y, z)$ attraverso la superficie $\Phi(u,v) = (u, v, 1-(u^2+v^2))$ definita sul Dominio $D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 \leq 1\}$

SOL

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1-(x^2+y^2) \end{cases}$$

$\psi_x = (1, 0, -2x)$

$\psi_y = (0, 1, -2y)$

$\psi_x \wedge \psi_y = (2x, 2y, 1)$

$$\Rightarrow \int_\Sigma F \cdot n = \int_D F(\psi) \cdot N_1 = \int_D \begin{pmatrix} x^4 \\ y \\ 1-(x^2+y^2) \end{pmatrix} \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

$$= \int_D 2x^5 + 2y^2 + 1 - (x^2+y^2) \, dx \, dy$$

in polari: $\begin{cases} x = \rho \cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\theta) & \theta \in [0, 2\pi] \end{cases}$

$$\Rightarrow \int_0^{2\pi} \int_0^1 2\rho^6 \cos^5(\theta) + 2\rho^3 \sin^2(\theta) + \rho - \rho^3 \, d\rho \, d\theta =$$

$$= \int^{2\pi} 2\cos^5(\theta) \frac{\rho^7}{7} + 2\sin^2(\theta) \frac{\rho^4}{4} + \frac{\rho^2}{2} - \frac{\rho^4}{4} \Big|^1 d\theta =$$

$$= \int_0^{2\pi} 2\cos^5(\theta)\,\frac{\rho^7}{7} + 2\sin^2(\theta)\,\frac{\rho^4}{4} + \frac{\rho^2}{2} - \frac{\rho^4}{4}\,\Big|_0^1 \, d\theta =$$

$$= \int_0^{2\pi} \frac{2}{7}\cos^5(\theta) + \frac{1}{2}\sin^2(\theta) + \frac{1}{2} - \frac{1}{4} \quad d\theta \quad =$$

$$\left(\cancel{\frac{2}{7}\left(\frac{5}{8}\sin(\theta) + \frac{5}{48}\sin(3\theta) + \frac{1}{80}\sin(5\theta)\right)} + \frac{1}{2}\left(\frac{1}{2}\left(\theta - \sin(\theta)\cos(\theta)\right) + \frac{1}{2}\theta - \frac{1}{4}\theta\right)\right)\Big|_0^{2\pi} =$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(2\pi\right)\right) + \pi - \frac{1}{4}\cdot 2\pi = \frac{\pi}{2} + \pi - \frac{\pi}{2} = \pi$$

$\blacksquare$

19. Prima prova mandata da Arianna

domenica 29 gennaio 2023 17:47

Es 1:

Si determini una $f(x)$ lineare (eventualmente affine) t.c. la forma differenziale

$$\omega = -\frac{x}{x^2+y^2} f\left(\sqrt{x^2+y^2}\right) dx + \frac{y}{\sqrt{x^2+y^2}}\, dy$$

sia localmente esatta in $\mathbb{R}^2 \setminus \{(0,0)\}$

Si calcoli un potenziale

SOL

Vediamo se è chiusa:

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$$

$$\frac{\partial}{\partial y} F_1 = \frac{x\left(\frac{y}{\sqrt{x^2+y^2}}\right)\cdot 2y}{x^2+y^2} f\left(\sqrt{x^2+y^2}\right) - \frac{x}{\sqrt{x^2+y^2}} f'\left(\sqrt{x^2+y^2}\right) \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial x} F_2 = -y \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{1}{x^2+y^2}$$

$\Rightarrow$

$$-\frac{\cancel{xy}}{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}} = \frac{\cancel{xy}}{\sqrt{x^2+y^2}\,\cancel{(x^2+y^2)}} f\left(\sqrt{x^2+y^2}\right) - \frac{\cancel{xy}\; f'\left(\sqrt{x^2+y^2}\right)}{\cancel{x^2+y^2}}$$

in polari: $\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \end{cases}$

$$\Rightarrow \quad -\frac{1}{\rho} = \frac{1}{\rho} f(\rho) - f'(\rho)$$

$$\Rightarrow \quad f'(\rho) = \frac{f(\rho)}{\rho} + \frac{1}{\rho}$$

$$\Rightarrow \dots \Rightarrow \quad f(\rho) = \dots = \rho c - 1 = \rho c$$

$$\Rightarrow \quad f(x) = xc - 1$$

Quindi

$$w = -\frac{x\left[c\sqrt{x^2+y^2} - 1\right]}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy$$

$$\Rightarrow \quad w = \frac{x}{\sqrt{x^2+y^2}} - xc \; dx + \frac{y}{\sqrt{x^2+y^2}} dy$$

$$U(x,y) = \int \frac{x}{\sqrt{x^2+y^2}} - xc \; dx = \sqrt{x^2+y^2} - c\frac{x^2}{2} + K(y)$$

$U_y = F_2$

$$\Rightarrow \quad \frac{y}{\sqrt{x^2+y^2}} + K'(y) = \frac{y}{\sqrt{x^2+y^2}} \quad \Rightarrow \quad K'(y) = 0 \quad \Rightarrow \quad K(y) = K$$

$$\Rightarrow \quad U(x,y) = \sqrt{x^2+y^2} - c\,\frac{x^2}{2} + K$$

$\blacksquare$

Es 2:

Calcolare

$$\iiint_C z\sqrt{1-y^2}\,dx\,dy\,dz$$

$$\int_{\partial D^+} (3x^2 - y)\,dx + (x + 4y^3)\,dy$$

dove

$C =$ cilindro circolare retto di altezza 1 che ha per asse l'asse z e per base il cerchio unitario centrato nell'origine

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -1 \le x \le 1 \;,\; |x| \le y \le \sqrt{2-x^2} \right\}$$

Sol

$$\iiint_C z\sqrt{1-y^2}\,dx\,dy\,dz$$

in coordinate cilindriche:

$$\begin{cases} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \\ z=z \end{cases} \qquad \begin{array}{l} \rho\in[0,1] \\ \theta\in[0,2\pi) \\ z=[0,1) \end{array}$$

$$\int_0^1\int_0^1\int_0^{2\pi} z\rho\sqrt{1-\rho^2\sin^2(\theta)}\,d\theta\,d\rho\,dz$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^1 \rho\sqrt{1-\rho^2\sin^2(\theta)}\,d\rho\,d\theta$$

$$-\frac{1}{3\sin^2(\theta)}\left(1-\rho^2\sin^2(\theta)\right)^{3/2} \rightarrow$$

$$=\frac{1}{2}\int_0^{2\pi}-\frac{1}{3\sin^2(\theta)}\left(1-\rho^2\sin^2(\theta)\right)^{3/2}\Big|_0^1\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}-\frac{\left(\cos^2(\theta)\right)^{3/2}}{3\sin^2(\theta)}+\frac{1}{3\sin^2(\theta)}\,d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1-\cos^3(\theta)}{3\sin^2(\theta)}\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\underbrace{\frac{1}{3\sin^2(\theta)}}_{\Downarrow}-\frac{\cos^3(\theta)}{\sin^2(\theta)}\,d\theta=\frac{1}{2}\int_0^{2\pi}-\frac{\cos^3(\theta)}{\sin^2(\theta)}\,d\theta$$

$$-\frac{1}{3}\cot(\theta)\Big|_0^{2\pi}=\infty$$

$$=\frac{1}{2}\int^{2\pi}-\frac{\cos(\theta)}{\sin^2(\theta)}+\cos(\theta)\,d\theta$$

$$-\frac{1}{3}\cot(\theta)\Big|_0 = \infty$$

$$= \frac{1}{2}\int_0^{\dots} \underbrace{-\frac{\cos(\theta)}{\sin^2(\theta)}}_{\frac{1}{\sin(\theta)}\Big|_0^{2\pi} = -\infty} + \underbrace{\cos(\theta)}_{\sin(\theta)\Big|_0^{2\pi} = 0}\, d\theta$$

$$= ???$$

2) $$\int_{\partial D^+} (3x^2-y)\,dx + (x+4y^3)\,dy$$

$$F = (3x^2-y,\ x+4y^3,\ 0)$$

$$\text{rot}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 3x^2-y & x+4y^3 & 0 \end{vmatrix} = (0,0,1-1) = (0,0,2)$$

$$\Rightarrow \int_D \text{rot}(F)\cdot n = \int_D \begin{pmatrix}0\\0\\2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \int_D 2\, d\sigma$$

$$= 2\int_{-1}^{1}\int_{|x|}^{\sqrt{2-x^2}} dy\, dx$$

in polari:
$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \end{cases}$$

$$\int^{\frac{3}{4}\pi} \int^{\sqrt{2}} \qquad \int^{3/4\,\pi} \int^{\sqrt{2}}$$

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \int_0^{\sqrt{2}} 2\rho \, d\rho \, d\theta = \int_{\frac{\pi}{4}}^{3/4\,\pi} \rho^2 \Big|_0^{\sqrt{2}} = 2 \cdot \frac{\pi}{2} = \pi$$

$\boxtimes$

Es 3:

Calcolare flusso di $F(x,y,z) = (x^4, y, z)$ attraverso $\varphi(u,v) = (u, v, 1-(u^2+v^2))$ con Dominio la palla unitaria di centro l'origine

Sol

$\varphi_u = (1, 0, -2u)$

$\varphi_v = (0, 1, -2v)$

$\varphi_u \wedge \varphi_v = (2u, 2v, 1)$

$$\Rightarrow \int_D F(\varphi) \cdot N = \int_D \begin{pmatrix} u^4 \\ v \\ 1-(u^2+v^2) \end{pmatrix} \begin{pmatrix} 2u \\ 2v \\ 1 \end{pmatrix} du\, dv$$

$$= \int_D 2u^5 + 2v^2 + 1 - (u^2+v^2) \, du \, dv$$

in polari:
$$\begin{cases} u = \rho \cos(\theta) & \rho \in [0,1] \\ v = \rho \sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$= \int_0^1 \int_0^{2\pi} 2\rho^6 \cos^5(\theta) + 2\rho^3 \sin^2(\theta) + \rho - \rho^3 \, d\theta \, d\rho = \dots = \frac{3}{2}\pi \quad \blacksquare$$

Es 0.1:

Si Determini $f(x)$ t.c.

$$\omega = -\frac{x}{\sqrt{x^2+y^2}} f\left(\sqrt{x^2+y^2}\right) dx + \frac{y}{\sqrt{x^2+y^2}} dy$$

sia localmente esatta in $\mathbb{R}^2 \setminus \{(0,0)\}$

Calcolare un potenziale

SOL

$$\frac{\partial}{\partial y} F_1 = \frac{-(-x)\frac{1}{2\sqrt{x^2+y^2}} \cdot 2y}{x^2+y^2} f\left(\sqrt{x^2+y^2}\right) + \frac{-x}{\sqrt{x^2+y^2}} f'\left(\sqrt{x^2+y^2}\right) \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} \cdot f(\sqrt{x^2+y^2}) - \frac{xy}{x^2+y^2} f'(\sqrt{x^2+y^2})$$

$$\frac{\partial}{\partial x} F_2 = \frac{-y \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x}{x^2+y^2} = -\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Dure bygde $\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$

$$\Rightarrow \frac{\cancel{xy}}{(x^2+y^2)\sqrt{x^2+y^2}} f(\sqrt{x^2+y^2}) - \frac{\cancel{xy}}{x^2+y^2} f'(\sqrt{x^2+y^2}) = -\frac{\cancel{xy}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{1}{(x^2+y^2)\sqrt{x^2+y^2}} \cdot f(\sqrt{x^2+y^2}) - \frac{1}{x^2+y^2} f'(\sqrt{x^2+y^2}) = -\frac{1}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{1}{x^2+y^2} f'(\sqrt{x^2+y^2}) - \frac{1}{(x^2+y^2)\sqrt{x^2+y^2}} f(\sqrt{x^2+y^2}) = \frac{1}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Se $z = \sqrt{x^2+y^2}$

$$\frac{1}{z^2} f'(z) - \frac{1}{z^3} f(z) = \frac{1}{z^3}$$

$\Rightarrow$

$$f'(z) - \underbrace{\frac{1}{z}}_{a(z)} f(z) = \underbrace{\frac{1}{z}}_{f(z)}$$

$$A(z) = \int a(z) = \int -\frac{1}{z}$$

$$= -\log(z)$$

$$f(z) = e^{-A(z)} \int e^{A(z)} f(z) + c\,e^{-A(z)}$$

$$= e^{\log(z)} \int e^{-\log(z)} \cdot \frac{1}{z} + c\,e^{\log(z)}$$

$$= z \int \frac{1}{z} \cdot \frac{1}{z} + c \cdot z = z \int \frac{1}{z^2} + z \cdot c$$

$$= z\left(-\frac{1}{z}\right) + z \cdot c$$

$$= zc - 1$$

$$f(z) = zc - 1$$

$\Downarrow$

$$f\left(\sqrt{x^2+y^2}\right) = c\sqrt{x^2+y^2} - 1$$

$\Rightarrow$

$$w = -\frac{x}{\sqrt{x^2+y^2}}\left(c\sqrt{x^2+y^2} - 1\right) dx + \frac{y}{\sqrt{x^2+y^2}}\, dy$$

$$= -cx + \frac{x}{\sqrt{x^2+y^2}}\, dx + \frac{y}{\sqrt{x^2+y^2}}\, dy$$

$$= -cx + \frac{x}{\sqrt{x^2+y^2}} \, dx + \frac{y}{\sqrt{x^2+y^2}} \, dy$$

$$U(x,y) = \int \frac{x}{\sqrt{x^2+y^2}} - cx \, dx =$$

$$= \sqrt{x^2+y^2} - c\frac{x^2}{2} + k(y)$$

$$U_y = F_2 \Rightarrow$$

$$\Rightarrow \frac{y}{\sqrt{x^2+y^2}} + K'(y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow K'(y) = 0 \Rightarrow K(y) = K$$

$$\Rightarrow$$

$$U(x,y) = \sqrt{x^2+y^2} - \frac{1}{2}cx^2 + K$$

$\blacksquare$

20. Terza prova mandata da Arianna

lunedì 30 gennaio 2023 11:02

ES 0.1 :

Determinare $f \in C^1(\mathbb{R})$ lineare affine t.c. la forma differenziale

$$w(x,y) = \left(6x^2y + 2y^3 + 2y\, f(x^2+y^2)\right)dx + \left(3y^2 - 2x\, f(x^2+y^2)\right)dy$$

sia esatta.

Calcolare un potenziale che si annulla in $(0,0)$

SOL

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$$

$$\frac{\partial}{\partial y} F_1 = 6x^2 + 6y^2 + 2\left(f(x^2+y^2) + y\, f'(x^2+y^2)\cdot 2y\right)$$

$$= 6x^2 + 6y^2 + 2f(x^2+y^2) + 4y^2 f'(x^2+y^2)$$

$$\frac{\partial}{\partial x} F_2 = -2\left(f(x^2+y^2) + x\, f'(x^2+y^2)\cdot 2x\right)$$

$$= -2f(x^2+y^2) - 4x^2 f'(x^2+y^2)$$

$\Rightarrow$

$$6x^2 + 6y^2 + 2f(x^2+y^2) + 4y^2 f'(x^2+y^2) = -2f(x^2+y^2) - 4x^2 f'(x^2+y^2)$$

$\Rightarrow$

$$6(x^2+y^2) + 4f(x^2+y^2) + 4(x^2+y^2) f'(x^2+y^2) = 0$$

$\Rightarrow$

$$4(x^2+y^2) f'(x^2+y^2) + 4f(x^2+y^2) = -6(x^2+y^2)$$

Se $z = x^2+y^2$

$$4z\, f'(z) + 4 f(z) = -6z$$

$\Rightarrow$

$$f'(z) + \frac{1}{z} f(z) = -\frac{3}{2}$$

Per comodità

$$\Rightarrow y' + \underbrace{\frac{1}{z}}_{a(z)} y = \underbrace{-\frac{3}{2}}_{f(z)}$$

$$A(z) = \int \frac{1}{z}\, dz = \log(z)$$

$$y(z) = e^{-A(z)} \int f(z)\, e^{A(z)} + c\, e^{-A(z)}$$

$$= e^{-\log(z)} \int -\frac{3}{2} e^{\log(z)}\, dz + c\, e^{-\log(z)}$$

$$= \frac{1}{z} \int -\frac{3}{2} z + c\, \frac{1}{z}$$

$$= \frac{1}{z} \left[-\frac{3}{2} \frac{z^2}{2} \right] + \frac{1}{z} c = -\frac{3}{4} z + \frac{1}{z} c$$

$$\Rightarrow f(x^2+y^2) = y(z) = -\frac{3}{4}(x^2+y^2) + \frac{1}{x^2+y^2} c$$

$\Rightarrow$

$$w(x,y) = \left(6x^2 y + 2y^3 + 2y\left(-\frac{3}{4}(x^2+y^2) + \frac{1}{x^2+y^2} c\right)\right) dx$$

$$+ \left(3y^2 - 2x\left(-\frac{3}{4}(x^2+y^2) + \frac{1}{x^2+y^2} c\right)\right) dy$$

$\Rightarrow$

$\Rightarrow$

$$w = \left(6x^2y + 2y^3 - \frac{3}{2}y\,(x^2+y^2) + \frac{2y}{x^2+y^2}c\right)\partial x +$$

$$+ \left(3y^2 + \frac{3}{2}x(x^2+y^2) - \frac{2x}{x^2+y^2}c\right)\partial y$$

$$U(x,y) = \int 6x^2y + 2y^3 - \frac{3}{2}yx^2 - \frac{3}{2}y^3 + \underbrace{\frac{2y}{x^2+y^2}c}_{2c\,\mathrm{arctg}\left(\frac{x}{y}\right)}\;\partial x =$$

$$= 6y\frac{x^3}{3} + 2y^3x - \frac{3}{2}y\frac{x^3}{3} - \frac{3}{2}y^3x + 2c\,\mathrm{arctg}\left(\frac{x}{y}\right) + c(y)$$

$$U_y = F_2$$

$\Rightarrow$

$$2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 + 2c\left(\frac{1}{1+\frac{x^2}{y^2}}\cdot x - \frac{1}{y^2}\right) + c'(y) = F_2$$

$\Rightarrow$

$$2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 + 2c\left(\frac{y^2}{x^2+y^2}x - \frac{1}{y^2}\right) + c'(y) = F_2$$

$\Rightarrow$

$$2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 + \frac{2cxy^2}{x^2+y^2} - \frac{2c}{y^2} + c'(y) = 3y^2 + \frac{3}{2}x\,(x^2+y^2) - \frac{2xc}{x^2+y^2}$$

$\Rightarrow$

$$\cancel{2x^3 - \frac{x^3}{2}} + \cancel{6xy^2 - \frac{9}{2}xy^2} + \frac{2cxy^2}{x^2+y^2} - \frac{2c}{y^2} + c'(y) = 3y^2 + \cancel{\frac{3}{2}x^3} + \cancel{\frac{3}{2}xy^2} - \frac{2xc}{x^2+y^2}$$

$\Rightarrow$

$$\frac{2cxy^2}{x^2+y^2} - \frac{2c}{y^2} + c'(y) = 3y^2 - \frac{2xc}{x^2+y^2}$$

$\Rightarrow$

$$c'(y) = 3y^2 - \frac{2xc}{x^2+y^2} - \frac{2cxy^2}{x^2+y^2} + \frac{2c}{y^2}$$

$$\Rightarrow c(\eta) = \int 3\eta^2 - \underbrace{\frac{2xc}{x^2+\eta^2}}_{\frac{2c}{x}\cdot \operatorname{arctg}\left(\frac{\eta}{x}\right)} - \underbrace{\frac{2cx\eta^2}{x^2+\eta^2}}_{2cx\,\eta\,\log(x^2)} + \frac{2c}{\eta^2}$$

$$= \eta^3 - \frac{2c}{x}\operatorname{arctg}\left(\frac{\eta}{x}\right) - (\ ) - \frac{2c}{\eta}$$

$$\int \frac{\eta^2}{x^2+\eta^2} = \int \underbrace{\eta}_{f(x)} \underbrace{\frac{\eta}{x^2+\eta^2}}_{g'(x)\to g(x)=\frac{1}{2}\log(x^2+\eta^2)}$$

$$= \eta\cdot\frac{1}{2}\log(x^2+\eta^2) - \int \frac{1}{2}\log(x^2+\eta^2)\, d\eta$$

$$= \frac{1}{2}\eta\log(x^2+\eta^2) - \frac{1}{2}\int \log(x^2+\eta^2)\, d\eta$$

$$= \frac{1}{2}\eta\log(x^2+\eta^2) - \frac{1}{2}\int \log(x^2)\cdot \underbrace{\log(\eta^2)}_{\eta\log(\eta^2)-2\eta}\, d\eta$$

$$= \frac{1}{2}\eta\log(x^2+\eta^2) - \frac{1}{2}\log(x^2)\left(\eta\log(\eta^2) - 2\eta\right) =$$

$$= \frac{1}{2}\eta\log(x^2+\eta^2) - \frac{1}{2}\eta\log(x^2+\eta^2) + \eta\log(x^2)$$

????

Es 0.2

Calcolare $\displaystyle\int_\Sigma \frac{1}{\eta^4}\, d\sigma$

dove $\Sigma = \left\{ \begin{pmatrix} x \\ \eta \\ z \end{pmatrix} \in \mathbb{R}^3 \,/\, \eta \in [1,2],\ \eta = \frac{1}{\sqrt{x^2+z^2}} \right\}$

Sol

Parametrizzo:

$$\psi(x,z) = \begin{cases} x = x \\ y = \frac{1}{\sqrt{x^2+z^2}} \\ z = z \end{cases}$$

$$\psi_x = \left(1, -\frac{\frac{1}{2\sqrt{x^2+z^2}}}{x^2+z^2} \cdot 2x, 0\right) = \left(1, -\frac{x}{(x^2+z^2)\sqrt{x^2+z^2}}, 0\right)$$

$$\psi_z = \left(0, -\frac{\frac{1}{2\sqrt{x^2+z^2}}}{x^2+z^2} \cdot 2z, 1\right) = \left(0, -\frac{z}{(x^2+z^2)\sqrt{x^2+z^2}}, 1\right)$$

$$\psi_x \wedge \psi_z = \left(-\frac{x}{(x^2+z^2)\sqrt{x^2+z^2}}, -1, -\frac{z}{(x^2+z^2)\sqrt{x^2+z^2}}\right)$$

$$\|\ldots\| = \sqrt{1 + \frac{x^2}{(x^2+z^2)^2(x^2+z^2)} + \frac{z^2}{(x^2+z^2)^3}} = \sqrt{1 + \frac{x^2+z^2}{(x^2+z^2)^3}}$$

$$= \sqrt{1 + \frac{1}{(x^2+z^2)^2}}$$

$$\int_\Sigma \frac{1}{y^4}\, d\sigma = \int (z^2+x^2)^2 \cdot \sqrt{1 + \frac{1}{(x^2+z^2)^2}}\, dx\, dz$$

$$1 < y < 2 \Rightarrow 1 < \frac{1}{\sqrt{x^2+z^2}} < 2$$

$$\Rightarrow 1 > \sqrt{x^2+z^2} > \frac{1}{2}$$

$$\Rightarrow 1 > x^2+z^2 > \frac{1}{4} \Rightarrow \frac{1}{4} < x^2+z^2 < 1$$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [\frac{1}{2}, 1] \\ z = \rho\sin(\theta) & \theta \in [0, 2\pi) \end{cases}$

$$\int_0^{2\pi}\int_{\frac{1}{2}}^{1} \rho^5 \sqrt{1+\frac{1}{\rho^4}}\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{1}{2}}^{1} \rho^5\sqrt{\frac{\rho^4+1}{\rho^4}} = \int_0^{2\pi}\int_{\frac{1}{2}}^{1} \rho^3\sqrt{\rho^4+1} = \int_0^{2\pi} \frac{1}{6}(\rho^4+1)^{3/2}\Big|_{\frac{1}{2}}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}(2)^{3/2} - \frac{1}{6}\left(\frac{1}{2^4}+1\right)^{3/2} d\theta =$$

$$= \int_0^{2\pi} \frac{1}{6}\cdot 2\sqrt{2} - \frac{1}{6}\left(\frac{1}{16}+1\right)^{3/2} d\theta =$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{3} - \frac{1}{6}\left(\frac{17}{16}\right)^{3/2} d\theta =$$

$$2\pi\left(\frac{\sqrt{2}}{3} - \frac{1}{6}\left(\frac{17}{16}\right)^{3/2}\right)$$

ES 0.3 :

Calcolare $\iiint_D 2z \, dx\, dy\, dz$

dove $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z>0,\ 0<x<2y+1,\ x^2+4y^2+z^2<1 \right\}$

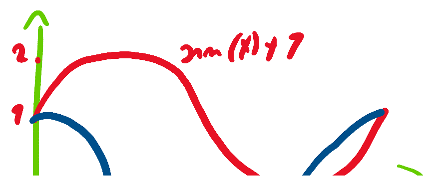
SOL

$x<2y+1 \Rightarrow \rho \sin(\psi)\cos(\theta) < \rho \sin(\psi)\sin(\theta)+1$

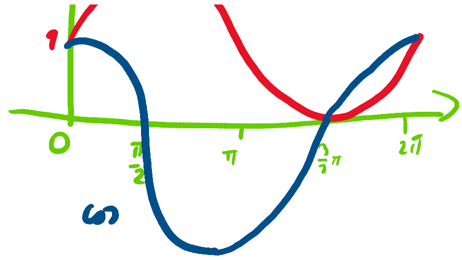
$\Rightarrow \cos(\theta) < \sin(\theta)+1 \Rightarrow \theta \in \left[0, \frac{3}{2}\pi\right]$

Parametrizzo l'ellissoide:

$$\begin{cases} x = \rho \sin(\psi)\cos(\theta) & \rho \in [0,1] \\ y = \frac{1}{2}\rho \sin(\psi)\sin(\theta) & \psi \in \left[0, \frac{\pi}{2}\right] \\ z = \rho\cos(\psi) & \theta \in \left[0, \frac{3}{2}\pi\right] \end{cases}$$



$$z = \rho \cos(\psi) \qquad \theta \in [0, \tfrac{3}{2}\pi]$$



$$\iiint_D 2z \, dx\, dy\, dz$$

$$\Rightarrow \iiint_D 2\rho \cos(\psi) \cdot \rho^2 \sin(\psi)$$

$$\Rightarrow \int_0^1 \int_0^{3/2\pi} \int_0^{\pi/2} 2\rho^3 \sin(\psi) \cos(\psi)\, d\psi\, d\theta\, d\rho$$

$$\Rightarrow \int_0^1 \int_0^{\frac{3}{2}\pi} 2\rho^3 \left. \frac{\sin^2(\psi)}{2} \right|_0^{\pi/2} d\theta\, d\rho =$$

$$= \int_0^1 \int_0^{\frac{3}{2}\pi} 2\rho^3 \left(\frac{1}{2} - 0\right) d\theta\, d\rho = \int_0^1 \int_0^{\frac{3}{2}\pi} \rho^3 \, d\theta\, d\rho =$$

$$= \int_0^1 \rho^3 \theta \Big|_0^{\frac{3}{2}\pi} d\rho = \int_0^1 \frac{3}{2}\pi \rho^3 = \frac{3}{2}\pi \int_0^1 \rho^3 \, d\rho =$$

$$= \frac{3}{2}\pi \cdot \left. \frac{\rho^4}{4} \right|_0^1 = \frac{3}{2}\pi \cdot \frac{1}{4} = \frac{3}{8}\pi \qquad \blacksquare$$

Es 0.1:

Determinare $f \in C^1(\mathbb{R})$ t.c.

$$\omega = \left(6x^2 y + 2y^3 + 2y\, f(x^2+y^2)\right) dx + \left(3y^2 - 2x\, f(x^2+y^2)\right) dy$$

sia esatta

Calcolare un potenziale

SOL

Sol

$$\frac{\partial}{\partial y} F_1 = 6x^2+6y^2+2f(x^2+y^2)+2y\, f'(x^2+y^2)\cdot 2y$$

$$\frac{\partial}{\partial x} F_2 = -2f(x^2+y^2)-2x\, f'(x^2+y^2)\cdot 2x$$

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2 \quad \Rightarrow$$

$\Rightarrow$
$$6x^2+6y^2+2f(x^2+y^2)+4y^2 f'(x^2+y^2) = -2f(x^2+y^2)-4x^2 f'(x^2+y^2)$$

$\Rightarrow$
$$6(x^2+y^2)+4f(x^2+y^2)+4y^2 f'(x^2+y^2)+4x^2 f'(x^2+y^2)=0$$

$\Rightarrow$
$$6(x^2+y^2)+4f(x^2+y^2)+4f'(x^2+y^2)\,(x^2+y^2)=0$$

Poniamo $z=x^2+y^2$

$\Rightarrow$
$$4f'(z)\cdot z+4f(z)+6z=0$$

$$\Rightarrow 4z\,f'(z)+4f(z)=-6z$$

$$\Rightarrow f'(z)+\underbrace{\frac{1}{z}}_{a(z)} f(z) = \underbrace{-\frac{3}{2}}_{f(z)}$$

$$A(z)=\int a(z)=\int \frac{1}{z}=\log(z)$$

$$f(z)=e^{-A(z)}\int e^{A(z)} f(z) + C e^{-A(z)}$$

$$=e^{-\log(z)}\int -\frac{3}{2} e^{\log(z)}\,dz + C e^{-\log(z)}$$

$$= \frac{1}{2} \cdot \int -\frac{3}{2} z \quad + \frac{1}{2} \cdot c$$

$$= \frac{1}{2} \cdot \left(-\frac{3}{2} \cdot \frac{z^2}{2}\right) + \frac{c}{z} \quad = -\frac{3}{4} z + \frac{c}{z}$$

$$f(x^2+y^2) = -\frac{3}{4}(x^2+y^2) + \frac{c}{x^2+y^2}$$

$\Rightarrow$

$$\omega = \left(6x^2y + 2y^3 + 2y\left(-\frac{3}{4}(x^2+y^2) + \frac{c}{x^2+y^2}\right)\right)dx + \left(3y^2 - 2x\left(-\frac{3}{4}(x^2+y^2) + \frac{c}{x^2+y^2}\right)\right)dy$$

$$\omega = \left(6x^2y + 2y^3 - \frac{3}{2}y(x^2+y^2) + \frac{2yc}{x^2+y^2}\right)dx + \left(3y^2 + \frac{3}{2}x(x^2+y^2) - \frac{2xc}{x^2+y^2}\right)dy$$

$$U(x,y) = \int 6x^2y + 2y^3 - \frac{3}{2}x^2y - \frac{3}{2}y^3 + \frac{2yc}{x^2+y^2}\, dx =$$

$$= 6\frac{x^3}{3}y + 2xy^3 - \frac{3}{2}\frac{x^3}{3}y - \frac{3}{2}xy^3 + 2c \cdot \operatorname{arctg}\left(\frac{x}{y}\right) + K(y)$$

$$U_y = F_2$$

$$U_y = 2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 + 2c\frac{1}{1+\frac{x^2}{y^2}} \cdot x \cdot \left(-\frac{1}{y^2}\right) + K'(y) =$$

$$= 2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 - 2c \cdot \frac{y^2}{y^2+x^2}\,\frac{x}{y^2} + K'(y) =$$

$$= 2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 - \frac{2cx}{x^2+y^2} + K'(y)$$

Segue che

$$2x^3 + 6xy^2 - \frac{x^3}{2} - \frac{9}{2}xy^2 - \cancel{\frac{2cx}{x^2+y^2}} + K'(y) = 3y^2 + \frac{3}{2}x(x^2+y^2) - \cancel{\frac{2xc}{x^2+y^2}}$$

$\Rightarrow$

$$\cancel{2x^3} + \cancel{6xy^2} - \cancel{\frac{x^3}{2}} - \cancel{\frac{9}{2}xy^2} + K'(y) = 3y^2 + \cancel{\frac{3}{2}x^3} + \cancel{\frac{3}{2}xy^2}$$

$\Rightarrow$

$$\cancel{2x^3} + \cancel{6xy^2} - \cancel{\frac{x^3}{2}} - \cancel{\frac{9}{2}xy^2} + K'(y) = 3y^2 + \cancel{\frac{3}{2}x^3} + \cancel{\frac{3}{2}xy^2}$$

$$\Rightarrow K'(y) = 3y^2 \quad \Rightarrow \quad K(y) = \int 3y^2 = y^3 + c$$

$\Rightarrow$

$$U(x,y) = 2x^3y + 2xy^3 - \frac{1}{2}x^3y - \frac{3}{2}xy^3 + 2c_1 \cdot \operatorname{arctg}\left(\frac{x}{y}\right) + y^3 + c_2$$

$$U(0,0) = 0$$

$$\Rightarrow y^3 + c_2 = 0 \quad \Rightarrow c_2 = -y^3$$

$\Rightarrow$

$$U(x,y) = 2x^3y + 2xy^3 - \frac{1}{2}x^3y - \frac{3}{2}xy^3 + 2c_1 \operatorname{arctg}\left(\frac{x}{y}\right)$$

∎

21. Seconda prova mandata da Arianna

lunedì 30 gennaio 2023 15:32

Es 0.3 :

Calcolare il flusso di $F(x,y,z) = \left(y e^{x+y} + \sin(z),\ \cos(z) - x e^{x+y},\ 2xy \log(x^2+y^2+1)\right)$

Uscente dalla superficie del solido

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ \middle|\ |y| \leq x \leq 2-|y|,\ 0 < z < x+y \right\}$$

SOL

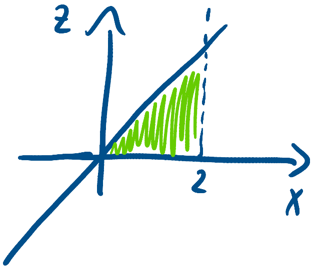
$$\operatorname{div}(F) = y e^{x+y} - x e^{x+y} + 0$$

$$= e^{x+y}(y-x)$$

Se $y=0$

$0 < x < 2$

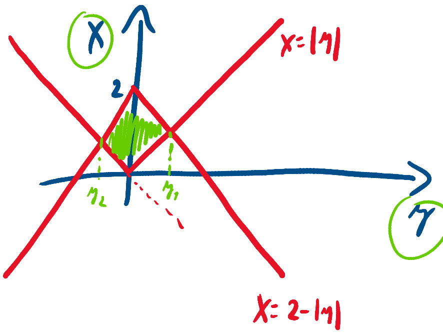
$0 < z < x$



Se $z=0$

$|y| < x < 2-|y|$

$0 \leq x+y \Rightarrow x \geq -y$



SONO INVERTITI GLI ASSI

$$\begin{cases} -y_1+2 = y_1 \Rightarrow 2y_1 = 2 \Rightarrow y_1 = 1 \\ -y_2 = y_2+2 \Rightarrow 2y_2 = -2 \Rightarrow y_2 = -1 \end{cases}$$

Riassumendo:

Sappiamo che $0 < z < x$ SO DOVE VARIA z

$D(x,y) = D_1 \cup D_2$

dove $D_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ \middle|\ y \in [-1,0],\ -y < x < y+2 \right\}$

$D_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ \middle|\ y \in [0,1],\ y < x < -y+2 \right\}$

$\Rightarrow$

$$\iiint_S \operatorname{div}(F) = \iiint_S e^{x+y}(y-x)\, dx\, dy\, dz =$$

$$= \iint_D \int_0^x e^{x+y}(y-x)\, dz\, dx\, dy =$$

$$= \iint_D e^{x+y}(y-x)\, z\Big|_0^x\, dx\, dy =$$

$$= \iint_D x e^{x+y}(y-x)\, dx\, dy =$$

$$= \iint_{D_1} x e^{x+y}(y-x)\, dx\, dy + \iint_{D_2} x e^{x+y}(y-x)\, dx\, dy$$

$$= \int_{-1}^{0}\int_{-y}^{y+2} x e^{x+y}(y-x)\, dx\, dy + \int_0^1 \int_y^{-y+2} x e^{x+y}(y-x)\, dx\, dy =$$

$$= \int_{-1}^{0}\int_{-y}^{y+2} \underbrace{xy e^{x+y}}_{\Downarrow\; xye^{x+y} - ye^{x+y}} - \underbrace{x^2 e^{x+y}}_{\Downarrow\; x^2e^{x+y} - 2xye^{x+y} + 2ye^{x+y}}\, dx\, dy + \int_0^1\int_y^{-y+2} xy e^{x+y} - x^2 e^{x+y}\, dx\, dy$$

$$= \int_{-1}^{0} xye^{x+y} - ye^{x+y} - x^2e^{x+y} - 2xye^{x+y} + 2ye^{x+y}\Big|_{-y}^{y+2}\, dy =$$

$$= \int_{-1}^{0} (y+2)y e^{y+2+y} - \left(-yy e^{-y+y}\right) - y e^{y+2+y} - \left(-y e^{-y+y}\right) - (y+2)^2 e^{y+2+y} - \left(-y^2 e^{-y+y}\right) - 2(y+2)y e^{2+y+y} - \left(-2(-y)y e^{-y+y}\right) + 2y e^{y+2+y} - 2y e^{-y+y}\, dy$$

$$= \int_{-1}^{0} (y+2)y e^{2+2y} \cancel{+y^2} - y e^{2+2y} + y - (y+2)^2 e^{2+2y} \cancel{+y^2} - 2(y+2)y e^{2+2y} \cancel{-2y^2} + 2y e^{2+2y} - 2y\, dy$$

$$= \int_{-1}^{0} -y(y+2) e^{2+2y} + y e^{2+2y} - y - (y+2)^2 e^{2+2y}\, dy$$

$$= \int_{-1}^{0} -y^2 e^{2+2y} - 2y e^{2+2y} + y e^{2+2y} - y - (y^2+4+2y) e^{2+2y}\, dy$$

$$= \int_{-1}^{0} -y^2 e^{2+2y} - y e^{2+2y} - y - y^2 e^{2+2y} - 4 e^{2+2y} - 2y e^{2+2y}\, dy$$

$$= \int_{-1}^{0} -\eta^2 e^{2+2\eta} - \eta e^{2+2\eta} - \eta - \eta^2 e^{2+2\eta} - 4e^{2+2\eta} - 2\eta e^{2+2\eta} \, d\eta$$

$$= \int_{-1}^{0} -2\eta^2 e^{2+2\eta} - 3\eta e^{2+2\eta} - \eta - 4e^{2+2\eta} \, d\eta$$

* da fonore *

22.

lunedì 30 gennaio 2023 17:17

ES 0.1 :

Determinare $f \in C^1(\mathbb{R})$ t.c. $f(0)=1$ e t.c.

$$w(x,y) = \frac{2xy\, f^2(x)}{1+f^2(x)}\,dx - \operatorname{arctg}(f(x))\,dy$$

sia esatta

Calcolare potenziale

SOL

$$\frac{\partial}{\partial y} F_1 = \frac{2x\, f^2(x)}{1+f^2(x)}$$

$$\frac{\partial}{\partial x} F_2 = -\frac{1}{1+f^2(x)}\, f'(x)$$

Deve essere $\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$

$\Rightarrow$

$$2x\, f^2(x) = -f'(x) \Rightarrow$$

$$\Rightarrow \quad f'(x) = -2x\, f^2(x)$$

$$\Rightarrow \quad f'(x) = -2x f^2(x)$$

$$y' = -2x\, y^2$$

$$\frac{dy}{dx} = -2x\, y^2$$

$$\frac{1}{y^2}\, dy = -2x\, dx$$

$$\int \frac{1}{y^2}\, dy = \int -2x\, dx$$

$$-\frac{1}{y} = -x^2 + c \quad \Rightarrow \quad \frac{1}{y} = x^2 + c$$

$$\Rightarrow \quad y = \frac{1}{x^2 + c}$$

$$\Rightarrow \quad f(x) = \frac{1}{x^2 + c}$$

$$f(0) = 1 \quad \Rightarrow \quad \frac{1}{c} = 1 \quad \Rightarrow \quad c = 1$$

$$\Rightarrow \quad f(x) = \frac{1}{x^2 + 1}$$

$$w = \frac{2xy\left(\frac{1}{x^2+1}\right)^2}{1+\left(\frac{1}{x^2+1}\right)^2}\,\partial x - \operatorname{arctg}\left(\frac{1}{1+x^2}\right)\partial y$$

$$w = \frac{2xy\cdot\frac{1}{x^4+1+2x^2}}{1+\frac{1}{x^4+1+2x^2}}\,\partial x - \operatorname{arctg}\left(\frac{1}{1+x^2}\right)\partial y$$

$$= \frac{2xy}{(1+x^2)+1}\,\partial x - \operatorname{arctg}\left(\frac{1}{x^2+1}\right)\partial y$$

$$U(x,y) = \int -\operatorname{arctg}\left(\frac{1}{x^2+1}\right)\partial y = -y\operatorname{arctg}\left(\frac{1}{x^2+1}\right) + K(x)$$

$$U_x = -y\,\frac{1}{1+\frac{1}{(x^2+1)^2}}\cdot\frac{-2x}{(1+x^2)^2} + K'(x)$$

$$= \frac{2xy}{(x^2+1)^2+1} + K'(x) = \frac{2xy}{(1+x^2)+1}$$

$$\Rightarrow \quad K'(x) = 0 \quad \Rightarrow \quad K(x) = K$$

$\Rightarrow$

$$U(x,y) = -y\operatorname{arctg}\left(\frac{1}{\cdots}\right) + K$$

$\Rightarrow$

$$U(x,y) = -y \, \mathrm{arctg}\left(\frac{1}{1+x^2}\right) + K$$

$$U(1,0) = 0$$

$$\Rightarrow \quad K = 0$$

$$\Rightarrow U(x,y) = -y \, \mathrm{arctg}\left(\frac{1}{1+x^2}\right)$$

$\blacksquare$

Es 3:

Calcolare flusso di

$$F(x,y,z) = \left(y\, e^{x+y} + \sin(z), \ \cos(z) - x\, e^{x+y}, \ 2xy \log(x^2+y^2+1)\right)$$

Uscente da

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ \middle| \ |y| \leq x \leq 2 - |y|, \quad 0 \leq z \leq x+y \right\}$$

Sol

$$\mathrm{div}(F) = y\, e^{x+y} - x\, e^{x+y} = e^{x+y}(y-x)$$

$$\iiint_S e^{x+y}(y-x) \, dx\,dy\,dz$$

$$= \int\int\int^{x+y} e^{x+y}(y-x) \, dz \, dx\,dy =$$

$$= \int_R \int_0^{x+y} e^{x+y}(y-x)\, dz\, dx\, dy =$$

$$= \int_R (y-x)(y+x)\, e^{x+y}\, dx\, dy$$

1

$$= \int_0^1 \int_{-x}^{x} (y^2-x^2)\, e^{x+y}\, dy\, dx$$

$$= \int_0^1 \int_{-x}^{x} y^2 e^{x+y} - x^2 e^{x+y}\, dy\, dx$$

23. Prova di A3 terzo foglio

martedì 31 gennaio 2023 16:39

ES 1

SOL

$$\frac{\partial}{\partial y} F_1 = 6x^2 + 6y^2 + 2f(x^2+y^2) + 2y\, f'(x^2+y^2) \cdot 2y$$

$$\frac{\partial}{\partial x} F_2 = -2 f(x^2+y^2) - 2x\, f'(x^2+y^2) \cdot 2x$$

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$$

$\Rightarrow$

$$6x^2 + 6y^2 + 2f(x^2+y^2) + 4y^2 f'(x^2+y^2) = -2f(x^2+y^2) - 4x^2 f'(x^2+y^2)$$

$\Rightarrow$